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Asteroseismology of white dwarf stars

D E Winget
McDonald Observatory and Department of Astronomy, University of Texas, Austin, TX 78712, USA

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Abstract. An understanding of the white dwarf stars is central to much of astrophysics, from the structure and evolution of stars to the age and history of the large ensembles of stars that we call galaxies. They are of great potential interest from the standpoint of physics as well, because they offer a chance to study matter under extreme conditions not yet attainable in the laboratory. White dwarfs are the simplest kinds of star that we know of, yet they are still very complicated. There are a number of questions that we must answer in order to understand them and the physics that governs their behaviour. Here I describe how we can use asteroseismology to answer these questions. I give a quick introduction to the field, summarize what has been done, and describe what looks promising for the future.

1. Physical context

Let me first give a physical context with motivations. Clearly it has been said for a long time—and I learned to say these words at the feet of my thesis advisor, Hugh Van Horn, who must have learned to say them at the feet of his thesis advisor, Ed Salpeter: white dwarfs are cosmic laboratories for studying the state of matter in extreme conditions of temperature and pressure. So we see they are quite relevant for the seminar to which this Special Issue is devoted.

For the hot pre-white-dwarf stars the neutrino luminosity can reach ten times the photon luminosity. Many types of neutrino are expected theoretically, but throughout most of the range where neutrino energy loss is important, plasmon neutrinos dominate the energy-loss rates (see Itoh *et al* (1996) for the most up-to-date computations of these rates). So these stars provide an excellent way, currently the only way, to study the production rate of plasmon neutrinos.

In contrast, for the coolest white dwarf stars the ratio of the Coulomb energy to the kinetic energy exceeds one hundred and eighty or so for the ions and you expect the ions to crystallize. It has now been more than 38 years since Ed Salpeter (Salpeter 1961), along with others independently (Kirzhnits 1960, Abrikosov 1961), made the first predictions of ion crystallization in the interior of white dwarf stars. So for nearly 40 years there has been no way to test Salpeter's theoretical prediction.

In figure 1 we have a chance to examine the large swath that the white dwarfs cut through the equation-of-state (EOS) plane. The upper boundary is provided by a $0.6M_\odot$ model (Kawaler 1986) with a surface temperature representative of the hottest pre-white-dwarf stars; the lower bound is a cool $1.2M_\odot$ model (Montgomery 1998) characteristic of the coolest white dwarf stars. The shaded region certainly covers much of the domain that we would consider to be extreme conditions, underscoring the point that the white dwarf stars have much to teach us.

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Note the temperature inversion in the centre of the hot model caused by neutrino cooling. The complicating effects of partial ionization are important in the surface layers, and in the coolest stars, some with nearly pure hydrogen surface layers, molecular hydrogen is important. In the deep interiors, electron degeneracy pressure supports the star, the electrons becoming relativistic in the massive cooler models.

In white dwarf stars the mechanical and thermal properties are well separated. While the electrons provide the pressure support, the ions provide the reservoir of heat which is the only significant energy source available (see Van Horn (1971) for a nice discussion). In the hot models the ions form a classical ideal gas, becoming a Coulomb fluid as they cool, and eventually a crystalline solid. Energy transport in the interior is dominated first by neutrino energy transport, and then by conduction, and in the outer layers by radiation, and in some temperature regions by the convection associated with partial ionization. I recommend you once again to the detailed discussion of all of this and more in Jordi Isern's paper in this Special Issue (Isern 1998a). With this in mind, we glance again at figure 1, and realize that we must understand the entire shaded region, if we are to model the white dwarf stars. Or looked at in another, more optimistic way, this is the part of the EOS plane that we *can* study by using the white dwarf stars.

2. Introduction to asteroseismology

What good is asteroseismology? Well, the curse of astronomy is that as experimentalists, we are unable to perform manipulations on the objects that we like to study, so we are reduced to looking at the light that we see from these objects, and observing only that

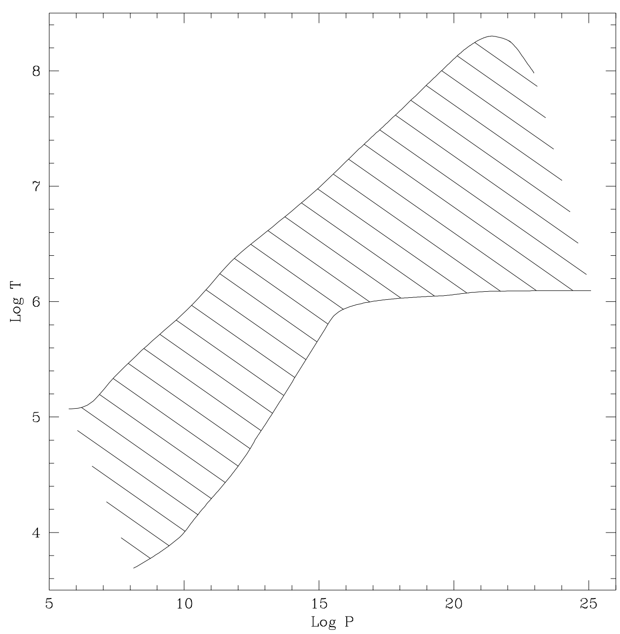

Figure 1. The region of the P–T plane spanned by the white dwarf star models. All quantities are in cgs units. The extreme boundaries are approximately marked at the upper left by a $0.6M_{\odot}$ model appropriate to the hot pre-white-dwarf stars and at the lower right boundary by a $1.2M_{\odot}$ model representing the coolest observed white dwarf stars.

which Nature chooses to show us. Of course we only see the photospheres, the thin outer skins. Asteroseismology gives us a way to look inside.

How do we do asteroseismology? Through an approach that we are all very familiar with as physicists: normal-mode analysis. There is an obvious analogy with the seismology of the Earth, from which the ideas for asteroseismology have been borrowed (for an elegant summary see the paper by S Yoshida in this Special Issue (Yoshida 1998)). Perhaps a more familiar analogy is that with atomic structure. We see from the mathematics why the problems are so analogous. Both are simple spherical-potential problems. The pulsation modes in the star are described in the same way as the energy levels of the atom.

2.1. Non-radial pulsations

In order to understand the oscillations that we actually observe, we need to consider the general class of non-radial spheroidal oscillations. We arrive at the form of the equations that we actually solve by perturbing the fluid equations, and keeping only the terms of lowest order. Non-linear effects are clearly important, and of increasing interest, but they are well beyond the scope of our present discussion. We assume a static, spherical equilibrium structure which is given by a theoretical evolutionary model. Because the surface gravity is high, $\log g \sim 8$, in cgs units, and rotation rates are typically of order days, the spherical approximation is quite good, and so we can expand our solutions in terms of spherical harmonics ($Y_{\ell m}$). Furthermore, we seek periodic solutions of the form

$$\xi(\boldsymbol{r}, t) = \xi(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\sigma t}.$$

The most relevant parameters which describe the equilibrium model are the Brunt–Väisälä frequency, which in the absence of chemical potential gradients is given by

$$N^2 \equiv -Ag = -g\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right]$$

which is just the difference between the actual and the adiabatic density gradients, and the Lamb, or acoustic, frequency given by

$$S_\ell^2 \equiv \frac{\ell(\ell+1)}{r^2}\frac{\Gamma_1 P}{\rho} = \frac{\ell(\ell+1)}{r^2}v_s^2$$

where v_s is the local speed of sound and all of the other quantities have their usual meanings.

We can gain a great deal of physical insight into the oscillations, and how they sample the star, through a sort of local analysis (see, for example, Unno *et al* 1989). If we assume a radial dependence of the form $\mathrm{e}^{\mathrm{i}k_r r}$, and wavelengths short compared to the relevant scale-heights for the physical quantities, we arrive at a local dispersion relation (LDR) of the form

$$k_r^2 = \frac{k_h^2}{\sigma^2 S_\ell^2}(\sigma^2 - N^2)(\sigma^2 - S_\ell^2)$$

where we have defined a horizontal wavenumber,

$$k_h^2 \equiv \frac{\ell(\ell+1)}{r^2}\frac{S_\ell^2}{v_s^2}$$

such that the total wavenumber is

$$k^2 \equiv k_k^2 + k_r^2.$$

The LDR allows us to see how the two characteristic frequencies determine the non-radial pulsation properties of the star. In order for a given mode to be locally propagating,

k_r^2 must be positive, so from the above expression we see that this occurs only when the oscillation frequency is greater than both N and the S_ℓ, or is less than both.

Taking the limits of large and small frequencies, the LDR yields two physically distinct kinds of solution that represent the two principal classes of non-radial spheroidal modes:

(1) $\sigma \gg N^2, S_\ell^2$:

$$\sigma_p^2 \approx \frac{k^2}{k_h^2} S_\ell^2 = (k_r^2 + k_t^2) v_s^2$$

(2) $\sigma^2 \ll N^2, S_\ell^2$:

$$\sigma_g^2 \approx \frac{k_h^2}{k_r^2 + k_h^2} N^2.$$

The first class of solutions represent the p-modes, so called because pressure is the principal restoring force. Radial displacements are dominant, and for white dwarf stars these have timescales of seconds, too short to be the periods of the observed oscillations. Also, one would not expect to observe large radial displacements for such a high-gravity object as a white dwarf. The second class of modes are the g-modes, where gravity is the dominant restoring force. These have timescales of hundreds of seconds and longer, just like the observed oscillations in the white dwarf stars. Also, the motions are predominantly horizontal, along gravitational equipotential surfaces. These expressions also indicate that the frequencies of the g-modes decrease with increasing radial overtone number (shorter wavelength) with an accumulation point at zero frequency. We can construct an expression for g-mode frequencies from this by an integration over the star, and arrive at an expression

$$\sigma_{k,\ell,m} \approx \left\langle \frac{N^2 \ell(\ell+1)}{k^2 r^2} \right\rangle^{1/2} + \left(1 - \frac{C_k}{\ell(\ell+1)}\right) m\Omega.$$

I have included the second term on the RHS of the expression to indicate the effects of slow rotation on the frequencies—note that there is a somewhat similar effect due to magnetic fields which we do not explicitly include here. The effect is to break the spherical symmetry and make the azimuthal quantum numbers, m, non-degenerate. Here Ω is the rotation frequency and the constant C_k is a quantity depending on the eigenfunction but approximately equal to one in most cases. This last term produces the potential for fine structure, splitting each radial overtone into $2\ell + 1$ components. Observing the spacing between these components then allows us to measure the rotation rate of the star.

When the first term in the above expression is dominant (the slow-rotation, small-magnetic-field limit) we can also see another important feature of the g-mode frequencies: in a compositionally homogeneous star, we could expect the spacing of consecutive radial overtone g-modes, for a given spherical harmonic degree, ℓ, to be approximately uniform in period. Furthermore, the spacing depends on the integrated average of the N^2, and so is set primarily by the total mass of the star. Deviations from this uniform spacing give information about compositional stratification, and allow us to measure the masses of the different layers described by Jordi Isern.

We can use plots of the run of N^2 and S_ℓ^2 through the star to determine where modes of given frequencies will propagate. Such diagnostic plots are known as propagation diagrams, and contain essentially all of the information that we can obtain through asteroseismological analysis of the star. In a sense the ultimate of asteroseismological analysis is to use the distribution of observed frequencies to determine, empirically, the propagation diagram for the star. It is important to note that if this is known, all of the constitutive physics can be de-convolved—at least in principle. This kind of analysis is called seismological inversion, and has been used successfully so far on only one star, the Sun. This is because of the more limited number of modes observed for other stars, but there is hope for the future for this technique as applied to white dwarf stars—but more on this later. For now, we use mostly the forward technique which consists of matching the observed frequencies to models and trying to find the particular theoretical model which best fits the observed periods.

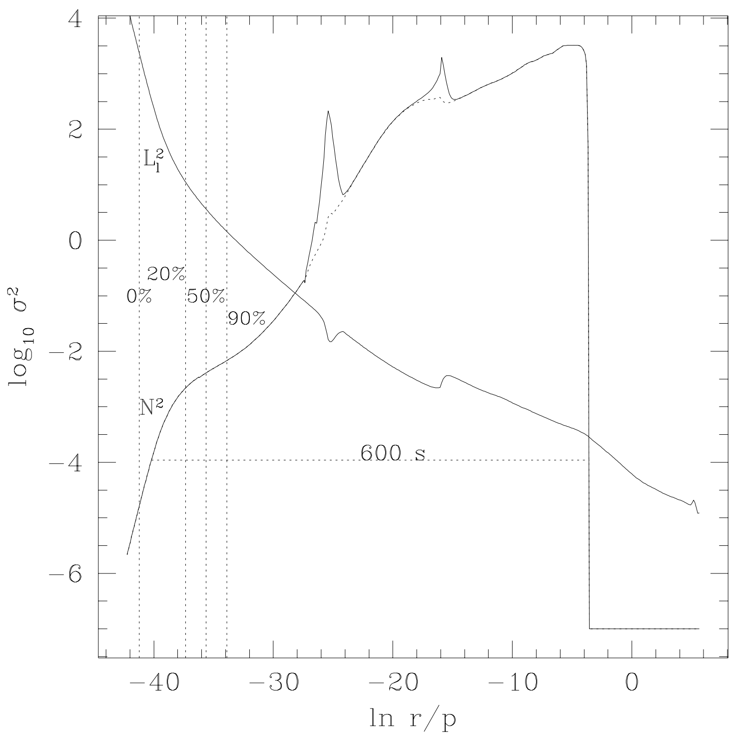

Figure 2. A propagation diagram for a model of a DAV white dwarf with mass $1.1M_{\odot}$. This illustrates the general behaviour of a propagation diagram for a white dwarf model and is appropriate to BPM 37093. The two solid curves show the Brunt–Väisälä frequency squared, N^2, and the square of the Lamb frequency, given by L_ℓ^2 (S_ℓ^2 in the text). The vertical lines indicate the mass fraction crystallized. The horizontal axis is the ratio of the radius to the pressure. This gives the appropriate resolution at both the centre and the surface of the model.

If we look at figure 2 we see an example of a propagation diagram derived from a theoretical white dwarf model. Studying it, we can learn a great deal about the pulsation properties of this (and similar) stars. Our dispersion relation shows us that the region of propagation for the g-modes is the region underneath both curves. We notice two things immediately. First that these are global oscillations, involving the whole star. Because of my choice of units of radius over pressure (to allow us to see features both near the surface and near the centre), the centre is off to the left here, but the point is that the modes penetrate deeply into the interior, so any information that we gain from looking at oscillations is information not just about the envelopes but about the interiors as well. Second, glancing at the vertical axis we can expect the g-modes to have periods much longer than tens of seconds and in the range of 100 to about 1000 seconds. These are just the timescales observed, as we shall see in the next section.

3. Current asteroseismology

All of the white dwarf pulsators seem to be otherwise normal white dwarf stars, so we may hope to apply what we learn through asteroseismological means to non-pulsating white dwarf stars as well. The oscillation of the stars on the white dwarf cooling track are all consistent with temperature variations caused by non-radial g-mode oscillations. The oscillations are detected as periodic variations in the mean intensity of the light.

Table 1. Observed properties of pulsating white dwarf stars.

Class	Spectra	$\log g$	$\log(L/L_\odot)$	T_e (K)	P (s) (range) Typical mode	Fractional amplitude (range) Typical mode
PNNV	He II, C IV Nebula	>6	3–4	$>100\,000$	($>$1000) 1500	($<$0.05–0.10) 0.01
DOV	He II, C IV, O VI Absorption with narrowing Empty core	7	2	$>100\,000$	(300–850) 500	($<$0.05–0.10) 0.01
DBV	He I Pure He absorption	8	−1.2	25 000	(100–1000) 500	($<$0.05–0.3) 0.02
HDAV	H Pure absorption	8	−2.8	12 000	(100–500)	($<$0.01–0.15) 0.01
CDAV	H Pure absorption	8	−2.8	11 000	(200–1500)	($<$0.15–0.3) 0.01

Nature has been kind in the distribution of the observed pulsators along the white dwarf cooling tracks, conveniently placing three distinct temperature regions of instability at nearly uniform intervals in the log of the surface temperature. I have listed the basic observed properties of the known classes of pulsators in table 1. In table 2, I summarize the theoretical properties. The V next to the spectral type in the class names indicates variable. I have divided the high-temperature instability strip into two classes: the hot DOV stars and the planetary nebula nuclei, or PNNV stars; this may be somewhat artificial, but the difference in their equilibrium properties and pulsation properties makes this a practical choice. Next along the cooling track we find the pulsating helium white dwarfs or DBV stars at around 25 000 K. The DAV stars are found at around 12 000 K and are the coolest known pulsators. Here I have subdivided them into the hot DAV (HDAV) stars and cool DAV (CDAV) stars on the basis of pulsation property differences, although they are contiguous in temperature and historically have been treated as one instability strip. Even the coolest known pulsators, the CDAV stars, are very hot compared to the temperature at which a typical $0.6M_\odot$ white dwarf will be expected to crystallize. So we were originally not optimistic about white dwarf asteroseismology telling us much about the detailed process of crystallization—but see below.

I have left off the newly discovered sdBV stars; these are subdwarf B stars pulsating in non-radial p-modes (Kilkenny *et al* 1997). These are extremely interesting objects whose pulsations were dramatically predicted in advance of their discovery by Charpinet *et al* (1996). Because these stars have not yet reached the white dwarf cooling track, because they are p-mode pulsators, and because of space limitations, we do not include them in our discussions here.

Looking briefly at the properties of these various pulsators we see that the periods are all typically 100 to 2000 seconds, and the amplitudes are similar as well. If we look at the theoretical models and ask what radial overtone numbers these periods correspond to, we see very high radial overtone numbers in the PNNV and DOV stars with progressively lower radial overtone numbers as we descend to cooler and cooler white dwarf stars, followed by a slight reversal of the trend in the CDAV stars. This may have to do with the specific properties of the driving mechanism, which is outside our scope.

In table 1 the amplitude range given is for the peak-to-peak variation of the total light intensity, while the typical period and amplitude entries correspond to typical individual modes. We find a broad range of stability of the frequencies and amplitudes of the modes. The PNNV and the CDAV stars seem to be very unstable, but we do see frequencies repeating over many observing seasons—indicating the presence of normal modes. The DOV and DBV stars seem to be intermediate in stability, and the HDAV stars exhibit extreme amplitude and frequency stability, with several stars stable to within observational limits over almost thirty years. These include the variable star G117-B15A which is the most stable optical variable in the sky with $\mathrm{d}P/\mathrm{d}t = (1.2 \pm 2.2) \times 10^{-15}$ s s^{-1} (Kepler *et al* 1998). This is consistent with zero and approaches the values expected from white dwarf evolution (cf. Bradley *et al* 1992). Such measurements will eventually provide important constraints on the interior chemical composition and evolutionary cooling times.

Table 2. A theoretical summary of pulsating white dwarf stars.

Class	Driving mechanism	ℓ	k ($\ell = 1$)
PNNV	C, O partial ionization, nuclear?	1–4	40–100
DOV	C, O partial ionization, nuclear?	1–4	20–40
DBV	He partial ionization	1–4	10–20
HDAV	H partial ionization	1–4	1–10
CDAV	H partial ionization	1–4	10–20

Now, all of these variables are multi-periodic. Some have greater than of the order of 100 modes so this is an ideal place to do asteroseismology. You are not restricted to one single frequency or two or three, typically. This is both a blessing and a curse, because how do you deal with resolving this number of frequencies in a light curve? The answer is something like the Whole Earth Telescope.

3.1. Brute-force asteroseismology with the Whole Earth Telescope

The Whole Earth Telescope (WET) was conceived by R E Nather (see Nather *et al* (1990) for a description of the WET as an instrument, and see Nather (1993) for a history). It is a group of scientists completely uninterested in politics and bureaucracy, simply talking to each other directly, organizing themselves on the basis of common scientific interests to observe a pre-selected target about twice a year from most of the major optical observatories on the planet. More than 50 scientists in more than 14 countries routinely participate in the Whole Earth Telescope runs, in order to obtain the kind of data that are necessary to understand the oscillation properties. The idea is to avoid the gaps in the data that would make disentangling all of the individual frequencies present impossible.

The WET observations produce a light curve after the data from each site are reduced and matched. Usually this is done in real time as the data come in from the sites. One takes the Fourier transform of the light curve and produces what Ed Nather is fond of calling a high-resolution temporal spectroscopy. From this set of individual peaks in the transform, we try to make mode identifications—in other words, finding unique values for the spherical harmonic degree, ℓ, the azimuthal quantum number, m, and also—if possible—-the radial overtone number.

Next we apply the tools of asteroseismology described above. The mean period spacing between effective radial overtones tells us the mass. Deviations from the mean period spacing allow us to map out the surface composition structure, including the surface layer masses. In principle, with enough modes, we should be able to determine the shape of the transition zones between layers of different chemical compositions, and even the core composition. Matching the luminosity of the star also allows us to determine the asteroseismological distance—typically considerably more accurate than parallax distances. For the brightest of the variable stars, at luminosities of $(10^2–10^4)L/L_{\odot}$, it gives us the *only* distance measurement, and makes the objects of interest for calibrating the cosmic distance scale.

Now, from the frequency splitting of each of the radial overtone multiplets, we can deduce rotation rates, and determine magnetic field strengths, and even the relative alignment of the magnetic and rotation axes.

3.2. Results for PG 1159 and GD 358 as proof of the concept

Light curves which yield rich sets of multiplets in their Fourier transforms are relatively easy to interpret asteroseismologically. For these stars the brute-force approach is very productive. In practice, one can calculate the period spacings of the central components of these multiplets and compare with the spacings calculated in a series of theoretical models. The fits are surprisingly good, but not exact at this stage. I believe that this represents deficiencies in our treatment of composition transition zones, but the fits are good enough to allow us to measure the mass of the star, the mass of the various layers of the star, and an accurate luminosity, effective temperature, and distance of the star (Winget *et al* 1991, Bradley and Winget 1994b, Kawaler and Bradley 1994).

A good example of WET data and the application of the brute-force technique is the first WET run on the DOV star PG1159-035 (see Winget *et al* (1991) for plots of the light curve, Fourier transform, and details of the analysis). Most of the power in the transform of the light curve is concentrated in a region between 200 s and 1000 s. You can clearly identify individual triplets of modes amongst the dominant power, and then at higher frequencies one can see quintuplets of modes. The ratio of the spacing of the triplets to the spacing in the quintuplets is 0.61 compared to the value of 0.60 for the ratio of $\ell = 1$ to $\ell = 2$ modes from the asymptotic expression given above. Furthermore, our asymptotic expression gives $\sqrt{3}$ for the ratio of the average spacing of $\ell = 1$ modes to that of $\ell = 2$ modes, and the ratio of the average period spacing between the central components of the triplets to that for the quintuplets is within a few per cent of this value. Not only do we see that the asymptotic theory works quite well for the high-overtone g-modes, but also we have two independent proofs of the ℓ-identifications. So we see the signature of rotational splitting in these multiplets, and we have many consecutive radial overtones and the evidence is only for $\ell = 1$ and $\ell = 2$ modes present in a star. Modes of higher spherical harmonic number are not seen. Why they are not is an open question in the field at present.

This very powerful technique gives you the most accurate stellar masses for field stars ever produced. The asteroseismological mass, for example, that we find for PG 1159 (the DOV star mentioned above) is $M_* = (0.59 \pm 0.01)M_{\odot}$, with internal errors a factor 0.3 smaller than that. The effective (surface) temperature is 136 000 K. The mass of the surface helium layer is not available in any other way, but the asteroseismological results are only consistent with about $4 \times 10^{-3} M_{\odot}$. The surface helium abundance is 0.27 by mass, and the distance based on the luminosity determination is 440 ± 40 pc.

We see a similar sort of thing looking at the WET data on a DBV white dwarf, GD 358 (see Winget *et al* 1994). Here we see only $\ell = 1$ modes, so we do not have independent tests showing that there is not some geometrical effect that is limiting it just to the central components of say $\ell = 2$ modes. Evidence for $\ell = 1$ comes from the period spacing and the asteroseismological matching of the mass and luminosity of the star. If we assume that the observed modes are $\ell = 2$ modes, the mass becomes unacceptable, the luminosity becomes unacceptable, and most convincingly the distance does not agree with the observed parallax (Bradley and Winget 1994b). With GD 358, as for the DOV stars, we again have sets of about 10 or 12 consecutive radial overtones that we observe multiplets for. We also see rich non-linear structure which I will not have space to describe, but we are beginning to make some progress in understanding it, and we are learning to use it as a signature for identifying the value of ℓ for pulsation modes (Brassard *et al* 1995).

On the basis of the best fits to the observed periods and their spacings, we obtain a measurement for the total mass of GD 358 of $M_* = (0.61 \pm 0.03) M_{\odot}$, a helium layer mass of $\log(M_{\mathrm{He}}/M_*) = -5.70^{+0.18}_{-0.30}$—much smaller than the value of $10^{-2} M_*$ predicted by stellar evolution theory (Iben 1989, D'Antona and Mazzitelli 1979), a transition zone thickness of about eight pressure scale-heights, and a luminosity of $\log(L_*/L_{\odot}) = -1.30^{+0.09}_{-0.12}$. This latter gives a distance of 42 ± 3 pc.

The asteroseismological determinations based on extensive WET observations give values in good agreement with the quantities determined using spectroscopic observations and parallaxes, where they exist, but the measurements of layer masses and transition profiles are not available from any other technique (for recent reviews of these aspects, see Kepler and Bradley (1995), and Kawaler (1998) and references therein). The results for PG 1159 and GD 358 demonstrate a proof of the concept for the brute-force method of asteroseismology

3.3. Asteroseismological mode identification for pulsators with few observed modes

Now, there are other stars which do not have so many modes observed and there are techniques that we can use to do asteroseismology with these objects. We do not have the space to do anything but briefly summarize these important techniques.

3.3.1. ℓ-identifications from non-linearities. WET observations of G117-B15A showed that each of the modes identified was split, except the dominant 215 s mode (Kepler 1993). Furthermore, each is split into several components, but this is clearly not the simple multiplet structure of the sort observed for PG 1159 and GD 358. Additional frequencies were detected with the WET, but these were clearly linear combination frequencies. Investigations by Brassard *et al* (1993) and Fontaine and Brassard (1994, and see also Fontaine *et al* 1996) using these data and independent large-telescope observations from the Canada–France–Hawaii Telescope show that all but three of the observed frequencies are linear combination frequencies, so there are only three independent pulsation modes present in the star. They use an analysis of the linear combination frequencies to demonstrate that the three modes must have the same values of ℓ and m (thus they must be different radial overtones).

This was an extremely important step, in that it was the first glimmer of hope that we might still be able to carry out ℓ-identification without large numbers of modes with consecutive values of the radial overtone number, n. Brassard *et al* (1995) elaborated on this technique, and showed its generality for DAV and DBV stars. They showed that the non-linearities associated with the inevitable transformation of linear temperature perturbations into luminosity (because $L \propto T^4$) leave a non-linear signature of ℓ. Thus the relative amplitudes of the harmonics and linear combination frequencies yield unambiguous ℓ-identifications. So far this technique seems to be applicable to the low-amplitude pulsators, but it has the advantage that it can be carried out using ground-based photometric observations.

3.3.2. Asteroseismology with the HST. A method of obtaining ℓ-identifications for individual modes has been pioneered by Robinson *et al* (1995). The idea is to use observations of changes in mode amplitudes as a function of the wavelength of observation. Robinson *et al* (1995) demonstrated that the much larger limb-darkening of white dwarf stars in the ultraviolet would lead to a very different (smaller) geometric cancellation effect in the ultraviolet. The idea is that differing ℓ-values can be distinguished on the basis of the behaviour of the amplitude as a function of wavelength. In particular, the ratio of the ultraviolet to the optical amplitude will be larger for $\ell = 2$ than for $\ell = 1$.

The analysis of Robinson *et al* (1995) used HST measurements of UV wavelengths. It established conclusively that the 215 s mode of G117-B15A is $l = 1$, confirming independently the earlier suggestion made by Brassard *et al* (1993) and Brassard and Fontaine (1994). This was also confirmed independently by Fontaine *et al* (1996) using the non-linear properties, and extended to the other two modes at 270.5 s and 304.1 s. The authors of the latter three papers were also able to conclude that $m = 0$ for all of the modes.

The ℓ-identification, and the fact that the density of modes at such short periods is quite small for any reasonable theoretical model, make it possible to use the value of the frequencies to determine the H-layer mass in this star, and constrain the total stellar mass. The various results due to Fontaine *et al* (1994) and Robinson *et al* (1995) have been summarized by Bradley (1996), and depend ultimately on whether or not the 215 s mode is the first or second radial overtone of the $\ell = 1$ g-modes. These two cases give, respectively, $M_H/M_* \sim 10^{-6}$ and $M_H/M_* \sim 10^{-4}$. As we see below, the work of Clemens, which implies that most if not all of the DAV stars have the same surface H-layer mass (except as regards variations associated with the total stellar mass), suggests that $n = 2$ may give the best fit, but this remains controversial.

3.3.3. Ensemble asteroseismology. The HDAV stars exhibit only a very small number of independent modes per star, and the CDAV stars, which often have a slightly larger set of modes, seem to be very unstable in amplitude, and possibly also slightly unstable in frequency. The problems are ameliorated slightly by the introduction of techniques using atmospheric properties to obtain ℓ-identifications as described above (and see Robinson *et al* (1995) and Fontaine *et al* (1996)).

Clemens (1993, 1994, 1995) and Kleinman ((1995a, b); see below) have come up with a new technique which allows us to make significant progress with these stars. The idea is to approach the HDAV stars and possibly the CDAV stars as an ensemble.

One of the principal results of Clemens' work is that although only a small handful of modes exist in any one of the HDAV stars, it is possible to combine the results for many stars, after correcting for differences in total stellar mass (cf. Bradley and Winget 1994a), and to show that they are consistent with a very narrow range of H-layer masses for all the DAV stars; the mode distributions, collectively, look like that of a single star, or at least the same basic compositional layer structure as is given by pre-white-dwarf evolutionary calculations (Iben and McDonald 1985).

These conclusions are still controversial, but are very consistent with the very homogeneous structure of the DAV instability strip: most if not all stars in the strip are variable—at least after removing the theoretically expected trend of blue-edge temperature with total stellar mass (Bergeron *et al* 1995, Clemens 1994, 1995, Kanaan 1996).

In contrast with the HDAV stars, the CDAV stars are very unstable in amplitude, and the power spectra have resisted even the heroic efforts spent on them—that is, until most recently. Scot Kleinman (Kleinman 1995a, b) and collaborators (Kleinman *et al* 1998a, b) have shown, through detailed analysis of the prototypical CDAV star G29-38, that the apparently erratically varying power spectra can be understood. Undaunted by transitions from high amplitude to low amplitude from season to season, attendant pulse shape changes, and tremendous variation sometimes from run to run (figure 3 of Kleinman *et al* 1998b), Kleinman showed that, on diligently keeping track of what modes appear and reappear, a clear set of stable, normal modes emerged. This work underscores the point that Nather often makes, that new techniques must be developed, at least at the beginning of any field, for each object observed.

Kleinman *et al* (1998b) show that there exists a recurring set of at least 19 distinct modes, not including multiplet components. Furthermore, they point out that it is likely that these are $\ell = 1$ modes of consecutive radial order. These conclusions are preliminary, so detailed asteroseismological conclusions would be premature. It is clear, however, that this work opens the door to asteroseismology for individual CDAV stars using similar techniques of extensive single-site observation. Interestingly, the WET is not the answer for these stars, but long-time-base single-site observations are.

4. The future of asteroseismology

Now a look to the future of asteroseismology. Determination of the basic physical parameters of the white dwarf stars is one of the most important problems in astronomy, with consequences for our understanding of stellar and galactic evolution (Isern 1998a, Winget 1997). As we have seen, asteroseismology is the way to measure these. Given that we can make these measurements, what do we need?

4.1. Isotopic asteroseismology

An interesting point about the He-layer mass in GD 358 is that the asteroseismological calculations suggest a mode-trapping source (i.e. a source of non-uniform period spacings) at a mass near $2 \times 10^{-6} M_*$; this may be not the He/C interface, but the signature of isotopic stratification. If the ^{3}He has separated gravitationally from the ^{4}He, then an isotopic mass fraction of 10^{-4} (not at all out of line with expected isotopic ratios) would produce a transition zone in the right place. This underscores the need to pay closer attention to possible separation of isotopes of the dominant species when modelling the asteroseismological data. Empirical determinations of isotopic mass fractions will place useful constraints on our theoretical understanding of nuclear burning processes both in stars and in the early universe.

4.2. More is better

4.2.1. More glass. Foremost in the search for the discovery of more pulsation modes in the known pulsating stars, we need large-telescope observations. This is the kind of observation that the University of Montreal group is doing in collaboration with the French group on the CFHT to detect observations of low-amplitudes modes. The goal is simply to get more modes in order to make a better job of all of the asteroseismological measurements.

Also, with large telescopes such as the Keck telescopes in Hawaii, or the HET telescope at the McDonald Observatory, it is possible to do time-resolved spectroscopy at high enough spectral and temporal resolution to explore the physics of the relative phases of maximum velocity and maximum light in the non-radial g-mode pulsations. This has just been done for the first time by Clemens *et al* (1998) and should lead to tests of new ideas about the driving of non-radial pulsations in the white dwarf stars (see for example Wu (1997), and Goldreich and Wu (1998) and references therein)

4.2.2. More pulsators. We need many more objects in each class. More than the proof of the concept, we need statistically significant samples of objects in each spectral class of white dwarf stars. For each spectral type we need better measurements for more objects. We need a statistically significant number of layer masses, total stellar masses, radii, and luminosities.

4.2.3. More classes of pulsators. In addition to more objects in the known classes, it would be very useful if we could find more classes of objects. There is some progress on this front. Witness the pulsating subdwarf B (sdBV) stars which were theoretically predicted to pulsate prior to their discovery, as described above. A remarkable triumph for the University of Montreal group, the sdBV stars are in fact a new class which have yet to be studied in detail asteroseismologically. And of course listening to Gilles Chabrier talk (the paper appears in this Special Issue (Chabrier 1998a)), I quickly scribbled in brown dwarf stars because it is clear that there is a large opacity and possibly a substantial region of partial ionization (or partial dissociation in the molecular case). As a result, we may expect to find pulsations in the brown dwarfs. This would be an extremely exciting development.

4.3. Inversions and genetic algorithms

Naoki Itoh asked exactly how we test the input physics with asteroseismology. This is a fair question, and the truth is that the so-called forward approach of matching frequencies to models is quite primitive in this regard, and will only lead us very gradually to appreciate where our constitutive physics falls short. Perhaps a better way is to use a parametrized form for the run of N^2 and S_ℓ^2 to obtain the best family of solutions possible for a given observed distribution of frequencies using a genetic algorithm approach. This, or something analogous, is necessary to ensure that we have an objectively complete sample of the possible range of propagation diagrams. Then we must compare the family of solutions with our models. We plan to carry out this approach at the University of Texas as part of the PhD Thesis of Travis Metcalfe, and Naoki should be hearing from us soon as we identify the shortcomings and successes of the input physics. I am convinced that this approach will revolutionize white dwarf asteroseismology, just as the inversion approach has revolutionized helioseismology.

4.4. Interior crystallization and asteroseismology

Finally, I want to mention the suspected crystallizing object BPM 37093. It violates the rule that I mentioned before that there would not be any crystallization in the DAV instability strip. That is because most white dwarf stars were $0.6 M_\odot$ stars. This star happens to be 1.05 solar masses (Bergeron *et al* 1995), and, according to our current ideas about crystallization, should be roughly 80% crystallized (Winget *et al* 1997) for a standard C/O mixed-core composition. Winget *et al* (1997) show further that, even if it is pure carbon in the centre, which is probably an unreasonable assumption, then the centre should be more that 50% crystallized. So here we have, nearly 40 years since the prediction of crystallization in the interior of white dwarf stars, a chance to test this asteroseismologically. To convince you that this can be done, we need to glance back at the propagation diagram for a model representing BPM 37093. Along with the Brunt–Väisälä frequency and the acoustic frequency for this model, we have sketched in the location of crystallization boundaries, for various possible crystallized mass fractions. So you can see for the g-modes in this star, near the value of the observed periods of about 600 seconds the propagation region is significantly affected by crystallization. Why? Because these are non-radial g-modes dominated by the horizontal displacement. Shear cannot be sustained in the crystalline interior, and, as a result, the g-modes would be stopped at the crystallization boundary. According to the calculations of Montgomery ((1998), and see Winget *et al* (1997)), the average period spacing in the region of 600 s is significantly affected by the crystallized mass fraction.

So, we just had a WET run on the objects, but first let us see theoretically what the effects are; then I will give you the results. The mean period spacing that you expect for a 20%, 80%, and 90% crystallized mass fraction increases by 10%, 20%, and 25%, respectively, relative to a model which is not crystallized. You can see in the domain around the expected 80% crystallized there is about a 20% increase in the period spacing over that of an uncrystallized model. The period spacing for $\ell = 1$ in an uncrystallized model is 26 s, so we would expect to find a spacing of 31 s if the theory is correct. So what is interesting is that we have a theoretical prediction.

What did we find when we actually looked at this star? The run finished less than a month before the meeting to which this Special Issue is dedicated, so we have only a quick reduction of the data at this point. We found eight frequencies, with six clearly representing different radial overtones of the same ℓ. They give a very surprising value of 18 s for the period spacing. What is happening? Is the theory completely wrong, and is the star close to the Chandrasekhar mass? Perhaps the answer is yes, but there is another possibility. Interestingly enough, if we accept that for some reason we are not seeing $\ell = 1$ modes which dominate in all of the other pulsators, but are in fact seeing the $\ell = 2$ modes almost exclusively, then we can scale using the asymptotic relationship that we looked at before, and we must multiply the observed spacing by roughly by $\sqrt{3}$ to obtain the $\ell = 1$ value. This gives a mean period spacing for the $\ell = 1$ modes of about 31 seconds. Interestingly enough, this puts it exactly at the 80% crystallized point. All of this is very preliminary, but it is enough to demonstrate that the determination of the ℓ-values of the modes in BPM 37093 is one of the most important problems in asteroseismology, and perhaps in all of astronomy. We will attempt to use the techniques described above to do exactly that with the Hubble Space Telescope.

5. Summary

So we have the possibility for the first time for testing the theory of white dwarf crystallization. I hope that I have convinced you that we can make determinations of basic white dwarf parameters using asteroseismology. Furthermore, that we need a statistically significant number of pulsators to observe for these to be useful constraints. We can explore the equation of state of matter at high densities and temperatures, measure plasmon neutrino energy production rates at the hot end of the white dwarf cooling tracks, study the Coulomb fluid domain in the middle, and study the region of the onset of crystallization and substantial crystallization at the very cool end.

We should also look for crystallization-driven pulsations in the cool white dwarf stars. As the crystallization boundary moves to the surface, we might expect that in the very coolest white dwarfs, particularly the most massive ones, it might be possible to get oscillations driven by the change in entropy associated with alternately melting and freezing the material at the crystal/fluid boundary and driving pulsations to observable amplitude. There is also the possibility of driving of a more conventional nature in the cool white dwarfs resulting from the large surface opacity observed for some of the stars. Given the potential importance of oscillations of either kind in these coolest white dwarf stars, this must be pursued both theoretically and observationally.

So, using asteroseismology we can dramatically improve our understanding of the largest two uncertainties in using the white dwarfs to get the age of the Galaxy: the distribution of surface layer masses and the effects of internal crystallization.

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